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Axisymmetric contact problems for an elastic layer resting on a rigid base with a Winkler type excavitation

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Abstract

Axisymmetric contact problems for an elastic layer pressed by a rigid sphere or by a rigid flat cylinder are considered. It is assumed that the layer rests on the rigid half-space with a near-boundary cylindrical excavitation which is filled with a deformable material. This material is modelled by a Winkler medium. The Hankel integral transforms are applied and the problems are reduced to the system of integral equations. The numerical analysis is performed to investigate the contact parameters and the deflexion in the excavitation zone. Results are presented in diagrams.

Keywords: Elastic layer; Contact problem; Excavitation; Winkler medium; Hankel transforms; Dual equations

1. Introduction

The axisymmetric contact problem for an elastic layer on a rigid smooth base was studied by Lebedev and Ufland (1958) and by Vorovich and Ustinov (1959). The boundary problems for a layer resting on the rigid base involving a cylindical hole were considered by Valov (1964), Low (1964), Zakorko (1974), Dhaliwal and Singh (1977), Grylitsky and Okrepky (1984) and Hara et al. (1990). The problems involving an underground excavitation have the great practical importance in the geotechnics and mining engineering.

In this paper two axisymmetric contact problems for the layer resting on the rigid base with cylindical excavitation filled with the Winkler medium are solved.

The geometry of the contact problems is shown in Fig. 1. The rigid indentor is pressed by a load P into the upper smooth boundary of an elastic homogeneous and isotropic layer of thickness H. Two important cases of geometry of punch are considered: the sphere of a radius R (Fig. 1b) and the flat cylinder of a radius l (Fig. 1c). The lower surface of the layer is supported by the rigid smooth base which is weaked by the near-boundary cylindical excavitation of a radius a. This excavitation is supposed to be filled by a

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Nomenclature

a	radius of the excavitation
$g(r)$	function describing the punch profile
H	thickness of the layer
$J_0(\cdot)$	Bessel function of the first kind
$K(\cdot)$	complete elliptical integral of the first kind
k	stiffness of the Winkler medium
l	radius of the contact zone
l_{H}	radius of the contact zone in the Hertz problem
P	load
P_{H}	load in the Hertz problem
$p(r)$	contact pressure
R	radius of the spherical indentor
r, z	cylindrical coordinates
u_r, u_z	elastic displacements
$w(r)$	deflexion in the excavitation zone
δ	centre displacement of the punch
$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$	harmonic operator in the cylindrical coordinates
$\phi(r,z)$	Airy function
$\kappa = l/a$	dimensionless ratio
$\kappa_1 = l/H$	dimensionless ratio
$\kappa_2 = a/H$	dimensionless ratio
λ, μ	Lame constants
v	Poisson's ratio
σ_r, σ_z, τ_{rz}	stress components
$\vartheta = \frac{(1-v)kH}{\mu}$	dimensionless stiffness of the Winkler medium
$*$	asterisks note the dimensionless values

deformable material which can be treated as the Winkler medium of stiffness k. The problems are considered to be axisymmetric.

2. Distributed load solution

Firstly we consider the boundary problem shown in Fig. 1a. In this problem it is assumed that the normal pressure σ_z on the upper surface of layer is given. The problem is described by the elasticity equations

$$\begin{aligned} 2(1-v)\frac{\partial \Theta}{\partial r} + (1-2v)\frac{\partial}{\partial z}\left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right) = 0 \\ 2(1-v)\frac{\partial \Theta}{\partial z} - (1-2v)\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_r}{\partial z} - r\frac{\partial u_z}{\partial r}\right) = 0 \end{aligned} \tag{1}$$

with the following boundary conditions

$$\tau_{rz}(r,H) = 0, \quad r \geqslant 0 \tag{2}$$

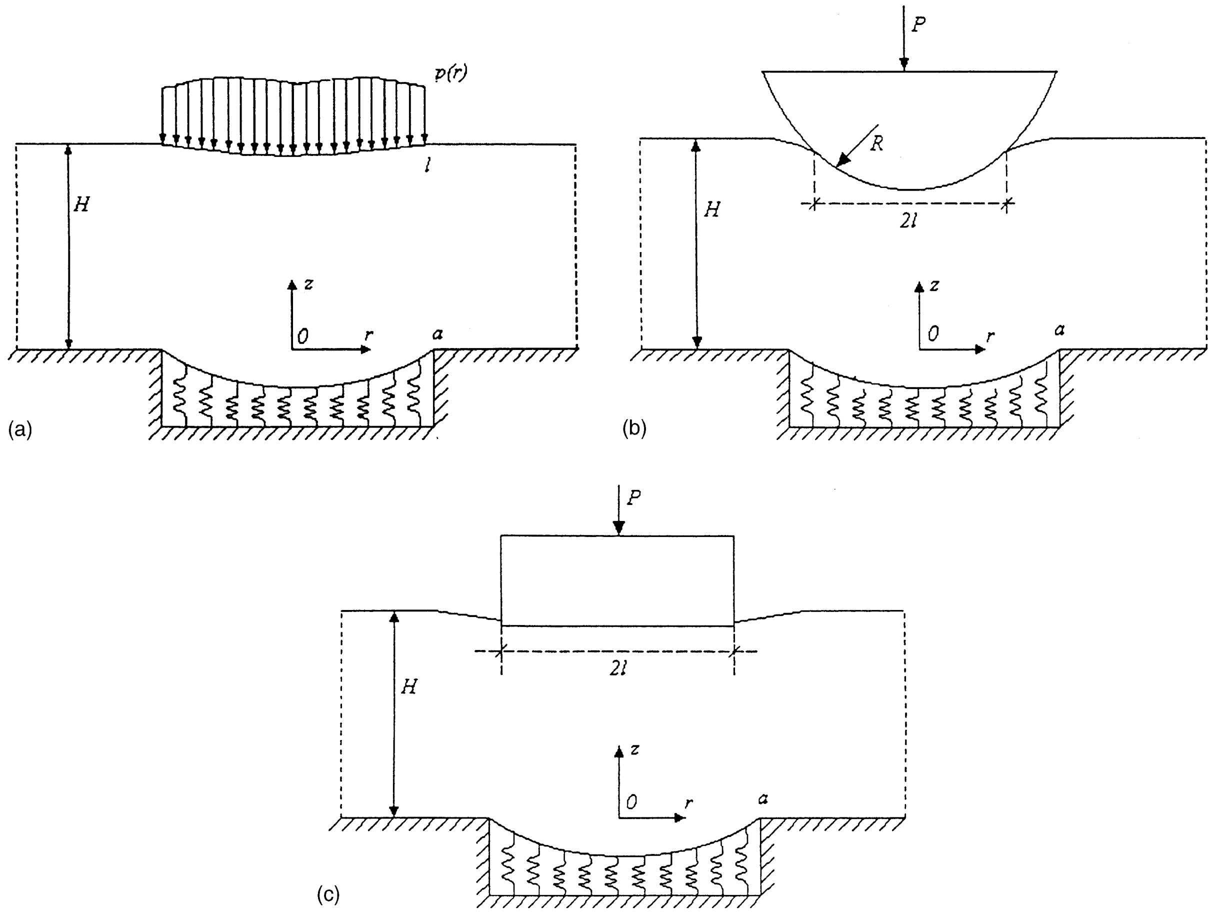

Fig. 1. Geometry of problems.

$$\sigma_z(r,H) = \begin{cases} -p(r), & 0 \leqslant r \leqslant l \\ 0, & r > l \end{cases} \tag{3}$$

$$\tau_{rz}(r,0) = 0, \quad r \geqslant 0 \tag{4}$$

$$u_z(r,0) = 0, \quad r > a \tag{5}$$

$$\sigma_z(r,0) = ku_z(r,0), \quad 0 \leqslant r \leqslant a \tag{6}$$

where the notation

$$\Theta = \frac{\partial u_r}{\partial r} + \frac{1}{r} u_r + \frac{\partial u_z}{\partial z}$$

is introduced.

The axisymmetric problem formulated above was solved using the Airy function $\phi(r,z)$ which satisfies the biharmonic equation

$$\Delta\Delta\phi(r,z) = 0 \tag{7}$$

Knowing this function, the displacements and stresses in the layer can be calculated by formulae (Sneddon, 1951)

$$
\begin{aligned}
u_r(r,z) &= -\frac{1}{1-2v}\frac{\partial^2\phi(r,z)}{\partial r\,\partial z} \\
u_z(r,z) &= \frac{2(1-v)}{1-2v}\Delta\phi(r,z) - \frac{1}{1-2v}\frac{\partial^2\phi(r,z)}{\partial r\,\partial z} \\
\sigma_z(r,z) &= (3\lambda+4\mu)\frac{\partial}{\partial z}\Delta\phi(r,z) - 2(\lambda+\mu)\frac{\partial^3\phi(r,z)}{\partial z^3} \\
\tau_{rz}(r,z) &= (\lambda+2\mu)\frac{\partial}{\partial r}\Delta\phi(r,z) - 2(\lambda+\mu)\frac{\partial^3\phi(r,z)}{\partial r\partial z^2}
\end{aligned}
\tag{8}
$$

The solution of Eq. (7) in the layer can be obtained by the Hankel transforms method and the function $\phi(r,z)$ is presented in the form

$$
\phi(r,z) = \int_0^\infty \{[A(\alpha)+\alpha z B(\alpha)]\cosh(\alpha z) + [C(\alpha)+\alpha z D(\alpha)]\sinh(\alpha z)\}J_0(\alpha r)\,\mathrm{d}\alpha \tag{9}
$$

where the functions $A(\alpha)$, $B(\alpha)$, $C(\alpha)$, $D(\alpha)$ are unknown and can be determinated from the boundary conditions (2)–(6).

Note that the boundary conditions (5) and (6) on the lower surface of the layer is of a mixed type. This fact does not permit to form a closed system of four equations for the functions $A(\alpha)$, $B(\alpha)$, $C(\alpha)$, $D(\alpha)$. Satisfying with the help of formulae (8) and (9) the boundary conditions (2)–(4) we arrive at three equations for the unknown functions. Solving these equations the functions $A(\alpha)$, $B(\alpha)$, $C(\alpha)$, $D(\alpha)$ can be represented by one function $\varphi(\alpha) \equiv \alpha^{-3}D(\alpha)$ which is unknown.

In result of the describing procedure the solution of the problem (1), (2)–(4) can be presented in the form (here solution is restricted to the normal stress and displacements on the surfaces of the layer which will be used in future analysis)

$$
\sigma_z(r,0) = -2(\lambda+\mu)\int_0^\infty \alpha F_1(\alpha)\varphi(\alpha)J_0(\alpha r)\,\mathrm{d}\alpha - \int_0^l r'p(r')S(r',r)\,\mathrm{d}r' \tag{10}
$$

$$
u_z(r,0) = \frac{2(1-v)}{1-2v}\int_0^\infty \varphi(\alpha)J_0(\alpha r)\,\mathrm{d}\alpha \tag{11}
$$

$$
u_z(r,H) = \frac{2(1-v)}{1-2v}\int_0^\infty F_2(\alpha)\varphi(\alpha)J_0(\alpha r)\,\mathrm{d}\alpha - \frac{1-v}{\mu}\int_0^l r'p(r')R_4(r',r)\,\mathrm{d}r' \tag{12}
$$

where

$$
\begin{aligned}
S(r',r) &= \int_0^\infty \alpha F_2(\alpha)J_0(\alpha r')J_0(\alpha r)\,\mathrm{d}\alpha \\
R_4(r',r) &= \int_0^\infty F_3(\alpha)J_0(\alpha r')J_0(\alpha r)\,\mathrm{d}\alpha \\
F_1(\alpha) &= \frac{\sinh^2(\alpha H) - \alpha^2H^2}{\sinh(\alpha H)\cosh(\alpha H)+\alpha H} \\
F_2(\alpha) &= \frac{\alpha H\cosh(\alpha H)+\sinh(\alpha H)}{\sinh(\alpha H)\cosh(\alpha H)+\alpha H} \\
F_3(\alpha) &= \frac{\sinh^2(\alpha H)}{\sinh(\alpha H)\cosh(\alpha H)+\alpha H}
\end{aligned}
\tag{13}
$$

The function $\varphi(\alpha)$ in the formulae (10)–(12) is unknown. For its determination the boundary conditions (5) and (6) must be used. Satisfying with the help of the presentations (10) and (11) these boundary conditions we arrive at the dual integral equation for the unknown function $\varphi(\alpha)$

$$\int_0^\infty \alpha\left[F_1(\alpha)+\frac{\vartheta}{\alpha H}\right]\varphi(\alpha)J_0(\alpha r)\,\mathrm{d}\alpha = -\frac{1}{2(\lambda+\mu)}\int_0^l r'p(r')S_1(r',r)\,\mathrm{d}r', \quad 0\leqslant r\leqslant a \tag{14}$$

$$\int_0^\infty \varphi(\alpha)J_0(\alpha r)\,\mathrm{d}\alpha = 0, \quad r>a \tag{15}$$

The solution of this dual equation is postulated in the form

$$\varphi(\alpha)=\int_0^a h(t)\sin(\alpha t)\,\mathrm{d}t \tag{16}$$

where $h(t)$ is a new unknown function. The form (16) satisfies automatically Eq. (15) and reduces Eq. (14) after some calculations to the Fredholm equation of the second kind for the function $h(t)$

$$h(t)-\frac{2}{\pi}\int_0^a h(t')R_1(t',t)\,\mathrm{d}t'+\frac{1}{\pi(\lambda+\mu)}\int_0^l r'p(r')R_2(r',t)\,\mathrm{d}r'=0, \quad 0\leqslant t\leqslant a \tag{17}$$

with the kernels

$$\begin{aligned} R_1(t',t) &= \int_0^\infty \left[1-F_1(\alpha)-\frac{\vartheta}{\alpha H}\right]\sin(\alpha t')\sin(\alpha t)\,\mathrm{d}\alpha \\ R_2(t',t) &= \int_0^\infty F_2(\alpha)J_0(\alpha t')\sin(\alpha t)\,\mathrm{d}\alpha \end{aligned} \tag{18}$$

If the distribution $p(t)$ is known, the function $h(t)$ can be obtained as the solution of Eq. (17). Substituting the presentation (16) into formulae (11) and (12) we obtain the deflexion in the excavitation zone

$$w(r)\equiv u_z(r,0)=\frac{2(1-\nu)}{1-2\nu}\int_r^a h(t)\frac{\mathrm{d}t}{\sqrt{t^2-r^2}}, \quad 0\leqslant r\leqslant a \tag{19}$$

and the normal displacements on the upper surface of the layer

$$u_z(r,H)=\frac{2(1-\nu)}{1-2\nu}\int_0^a h(t')R_3(t',r)\,\mathrm{d}t'-\frac{1-\nu}{\mu}\int_0^l r'p(r')R_4(r',r)\,\mathrm{d}r', \quad r\geqslant 0 \tag{20}$$

where

$$R_3(t',r)=R_2(r,t') \tag{21}$$

3. Integral equations of the contact problem

In the contact problem the normal stresses $\sigma_z(r,H)=-p(r)$ in the interaction area $0\leqslant r\leqslant l$ are not prescribed. They can be determinated satisfying the contact condition which in the case of a rigid punch is written as

$$u_z(r,H)=\delta-g(r), \quad 0\leqslant r\leqslant l \tag{22}$$

Substituting (22) into formulae (20) gives the following integral equation

$$\frac{2(1-\nu)}{1-2\nu}\int_0^a h(t')R_3(t',r)\,\mathrm{d}t' - \frac{1-\nu}{\mu}\int_0^l r'p(r')R_4(r',r)\,\mathrm{d}r' = \delta - g(r), \quad 0 \leqslant r \leqslant l \tag{23}$$

This equation and that (17) together with the equilibrium condition

$$2\pi\int_0^l r'p(r')\,\mathrm{d}r' = P \tag{24}$$

stand the closed system of integral equations of the contact problem. The functions $h(t)$, $p(r)$ are unknown.

The following particular case can be obtained from the system (17), (23) and (24):

(i) a layer resting on a smooth rigid base: $a = 0$ or $k \to \infty (\vartheta \to \infty)$, see Lebedev and Ufland (1958) and Vorovich and Ustinov (1959);
(ii) a layer resting on a smooth rigid base with the empty excavitation: $k = 0$ $(\vartheta = 0)$, see Valov (1964), Low (1964), Zakorko (1974), Dhaliwal and Singh (1977), Grylitsky and Okrepky (1984) and Hara et al. (1990);
(iii) a layer supported by a Winkler foundation: $a \to \infty$. Analogic plane problem was considered by Dempsey et al. (1990).

Introducing the dimensionless parameters

$$\begin{aligned}
&\tau = \frac{t}{a}, \quad \tau' = \frac{t'}{a}, \quad \rho = \frac{r}{l}, \quad \rho' = \frac{r'}{l}, \quad \beta = \alpha H, \quad p^*(\rho) = \frac{l^2}{P}p(r), \quad h^*(\tau) = \frac{\mu}{1-\nu}\frac{H}{P}h(t), \\
&\delta^* = \frac{\mu}{1-\nu}\frac{l}{P}\delta, \quad g^*(\rho) = \frac{\mu}{1-\nu}\frac{l}{P}g(r)
\end{aligned} \tag{25}$$

the system of governing integral equations (17), (23) and (24) can be rewritten

$$\begin{aligned}
&h^*(\tau) - \frac{2\kappa_2}{\pi}\int_0^1 h^*(\tau')R_1^*(\tau',\tau)\,\mathrm{d}\tau' + \frac{1}{\pi}\int_0^1 \rho' p^*(\rho')R_2^*(\rho',\tau)\,\mathrm{d}\rho' = 0, \quad 0 \leqslant \tau \leqslant 1 \\
&2\kappa_1\kappa_2\int_0^1 h^*(\tau')R_3^*(\tau',\rho)\,\mathrm{d}\tau' - \kappa_1\int_0^1 \rho' p^*(\rho')R_4^*(\rho',\rho)\,\mathrm{d}\rho' = \delta^* - g^*(\rho), \quad 0 \leqslant \rho \leqslant 1 \\
&2\pi\int_0^1 \rho' p^*(\rho')\,\mathrm{d}\rho' = 1
\end{aligned} \tag{26}$$

with the dimensionless kernels

$$\begin{aligned}
R_1^*(\tau',\tau) &= \int_0^\infty \left[1 - F_1^*(\beta) - \frac{\vartheta}{\beta}\right]\sin(\kappa_2\beta\tau')\sin(\kappa_2\beta\tau)\,\mathrm{d}\beta \\
R_2^*(\rho',\tau) &= \int_0^\infty F_2^*(\beta)J_0(\kappa_1\beta\rho')\sin(\kappa_2\beta\tau)\,\mathrm{d}\beta \\
R_3^*(\tau',\rho) &= R_2^*(\rho,\tau') \\
R_4^*(\rho',\rho) &= \int_0^\infty F_3^*(\beta)J_0(\kappa_1\beta\rho')J_0(\kappa_1\beta\rho)\,\mathrm{d}\beta \\
F_1^*(\beta) &= \frac{\sinh^2(\beta) - \beta^2}{\sinh(\beta)\cosh(\beta) + \beta} \\
F_2^*(\beta) &= \frac{\beta\cosh(\beta) + \sinh(\beta)}{\sinh(\beta)\cosh(\beta) + \beta} \\
F_3^*(\beta) &= \frac{\sinh^2(\beta)}{\sinh(\beta)\cosh(\beta) + \beta}
\end{aligned} \tag{27}$$

Taking into account the integrals (Gradshteyn and Ryzhik, 1965)

$$\int_0^\infty x^{-1}\sin(ax)\sin(bx)\,\mathrm{d}x = \frac{1}{4}\log\left[\frac{a+b}{a-b}\right]^2$$
$$\int_0^\infty J_0(ax)J_0(bx)\,\mathrm{d}x = \frac{2}{\pi}\begin{cases} a^{-1}K(ba^{-1}), & b<a \\ b^{-1}K(ab^{-1}), & b>a \end{cases}$$

the kernels $R_1^*(\tau',\tau)$ and $R_4^*(\rho',\rho)$ can be written in the forms displaying singular parts

$$\begin{aligned} R_1^*(\tau',\tau) &= -\frac{\vartheta}{4}\log\left[\frac{\tau'+\tau}{\tau-\tau}\right]^2 + \int_0^\infty \left[1-F_1^*(\beta)\right]\sin(\kappa_2\beta\tau')\sin(\kappa_2\beta\tau)\,\mathrm{d}\beta \\ R_4^*(\rho',\rho) &= \frac{2}{\pi\kappa_1\rho_1}K\left(\frac{\rho_2}{\rho_1}\right) + \int_0^\infty \left[F_3^*(\beta)-1\right]J_0(\kappa_1\beta\rho')J_0(\kappa_1\beta\rho)\,\mathrm{d}\beta \end{aligned} \tag{28}$$

where the notations $\rho_1 = \max(\rho',\rho)$, $\rho_2 = \min(\rho',\rho)$ are introduced.

Let us observe that the functions $1-F_1^*(\beta)$, $F_2^*(\beta)$ and $F_3^*(\beta)-1$ decay exponentially for $\beta\to\infty$. Thus these functions can be written in the forms of a finite exponential series

$$\begin{aligned} 1-F_1^*(\beta) &= \sum_{m=1}^{M_1} A_m^{(1)}\mathrm{e}^{-m\gamma_1\beta} \\ F_2^*(\beta) &= \sum_{m=1}^{M_2} A_m^{(2)}\mathrm{e}^{-m\gamma_2\beta} \\ F_3^*(\beta)-1 &= \sum_{m=1}^{M_3} A_m^{(3)}\mathrm{e}^{-m\gamma_3\beta} \end{aligned} \tag{29}$$

where the constants M_k, γ_k, $A_m^{(k)}$ are unknown. For their determination the squared error method was used. This approach, which was presented in paper Li and Dempsey (1990), is outlined in Appendix A.

Substituting the formulae (29) into (27) and (28) and using some integrals the kernels $R_1^*(\tau',\tau), R_2^*(\rho',\tau)$ and $R_4^*(\rho',\rho)$ can be written in the forms

$$\begin{aligned} R_1^*(\tau',\tau) &= -\frac{\vartheta}{4}\log\left[\frac{\tau+\tau}{\tau'-\tau}\right]^2 + \sum_{m=1}^{M_1} A_m^{(1)}\frac{2m\gamma_1\kappa_2^2\tau'\tau}{\left[m^2\gamma_1^2+(\tau'-\tau)^2\kappa_2^2\right]\left[m^2\gamma_1^2+(\tau'+\tau)^2\kappa_2^2\right]} \\ R_2^*(\rho',\tau) &= \sum_{m=1}^{M_2} A_m^{(2)}\frac{\sqrt{2}m\gamma_2\kappa_2\tau}{Z_1Z_2\sqrt{m^2\gamma_2^2+\rho'^2\kappa_1^2-\tau^2\kappa_2^2+Z_1Z_2}} \\ R_4^*(\rho',\rho) &= \frac{2}{\pi\kappa_1\rho_1}K\left(\frac{\rho_2}{\rho_1}\right) + \sum_{m=1}^{M_3} A_m^{(3)}\frac{1}{\sqrt{m^2\gamma_3^2+\kappa_1^2(\rho'+\rho)^2}}K\left(\frac{2\kappa_1\sqrt{\rho'\rho}}{\sqrt{m^2\gamma_3^2+\kappa_1^2(\rho'+\rho)^2}}\right) \\ Z_1 &= \sqrt{m^2\gamma_2^2+(\kappa_1\rho'+\kappa_2\tau)^2}, \quad Z_2 = \sqrt{m^2\gamma_2^2+(\kappa_1\rho'-\kappa_2\tau)^2} \end{aligned} \tag{30}$$

The dimensionless deflexion in the excavitation region introduced by formula

$$w^*(\rho) = \frac{2(1-v)a}{1-2v}\frac{\lambda+\mu}{P}w(r)$$

in the accordance with (19) can be written as

$$w^*(\rho) = \kappa_2 \int_\rho^1 h^*(\tau') \frac{d\tau'}{\sqrt{\tau'^2 - \rho^2}}, \quad 0 \leqslant \rho \leqslant 1 \tag{31}$$

4. Contact of a rigid sphere with a layer

First example concerns with the contact problem of a rigid sphere of the radius R with a layer supported by the base with an excavation (Fig. 1b). In this case the function $g(r)$ describing the punch geometry can be written as

$$g(r) = \frac{r^2}{2R}$$

or in the dimensionless form (25)

$$g^*(\rho) = \frac{3}{16} \frac{P_\mathrm{H}}{P} \left[\frac{l}{l_\mathrm{H}} \right]^3 \rho^2 \tag{32}$$

where P_H and l_H are, respectively, a load and contact radius in the Hertz problem, which are connected by formula (Johnson, 1987)

$$l_\mathrm{H}^3 = \frac{3}{8} \frac{(1-\nu)R}{\mu} P_\mathrm{H} \tag{33}$$

Substituting the formula (32) into (26, part 2) we arrive at the system of integral equations of the problem under consideration:

$$\begin{aligned} &h^*(\tau) - \frac{2\kappa_2}{\pi} \int_0^1 h^*(\tau') R_1^*(\tau', \tau) \, d\tau' + \frac{1}{\pi} \int_0^1 \rho' p^*(\rho') R_2^*(\rho', \tau) \, d\rho' = 0, \quad 0 \leqslant \tau \leqslant 1 \\ &2\kappa_1\kappa_2 \int_0^1 h^*(\tau') R_3^*(\tau', \rho) \, d\tau' - \kappa_1 \int_0^1 \rho' p^*(\rho') R_4^*(\rho', \rho) \, d\rho' = \delta^* - \frac{3}{16} \frac{P_\mathrm{H}}{P} \left[\frac{l}{l_\mathrm{H}} \right]^3 \rho^2, \quad 0 \leqslant \rho \leqslant 1 \\ &2\pi \int_0^1 \rho' p^*(\rho') \, d\rho' = 1 \end{aligned} \tag{34}$$

Note that the contact radius l is unknown. If the value P_H/P is given we can found the ratio l/l_H by solving Eq. (34) iteratively until the physical condition

$$p^*(1) = 0 \tag{35}$$

will be satisfied. Here we applied another simple way: the contact area is assumed to be equal to that in the Hertz problem (i.e. $l/l_\mathrm{H} = 1$), but the ratio P_H/P needed to obtain last relation is unknown. Then the system of integral equations (34) is sufficient to determine the distribution $h^*(\tau)$, $p^*(\rho)$ and values δ^*, P_H/P.

Discretizing the contact region $0 \leqslant \rho \leqslant 1$ and excavation zone $0 \leqslant \tau \leqslant 1$ into n pieces by points

$$\rho_i' = (i - 0.5)/n, \qquad \tau_i' = \rho_i', \quad i = 1, \ldots, n \tag{36}$$

respectively, the system (34) can be transformed to the equivalent system of linear algebraic equations

$$
\begin{aligned}
&h^*(\rho'_m) - \frac{2\kappa_2}{\pi}\sum_{k=1}^{n} h^*(\rho'_k)B^{(1)}_{km} + \frac{1}{\pi}\sum_{k=1}^{n} p^*(\rho'_k)B^{(2)}_{km} = 0, \quad m = 1,\ldots,n \\
&2\kappa_1\kappa_2\sum_{k=1}^{n} h^*(\rho'_k)B^{(3)}_{km} - \kappa_1\sum_{k=1}^{n} p^*(\rho'_k)B^{(4)}_{km} - \delta^* + \frac{3}{16}\frac{P_{\mathrm{H}}}{P}\rho_m^2 = 0, \quad m = 1,\ldots,n+1 \\
&\frac{\pi}{n^2}\sum_{k=1}^{n}(2k-1)p^*(\rho'_k) = 1
\end{aligned}
\tag{37}
$$

where

$$
\begin{aligned}
&\rho'_i = (i-1)/n, \quad i = 1,\ldots,n+1 \\
&B^{(1)}_{km} = \int_{\rho_k}^{\rho_{k+1}} R_1(\rho',\rho'_m)\,\mathrm{d}\rho', \quad B^{(2)}_{km} = \int_{\rho_k}^{\rho_{k+1}} \rho' R_2(\rho',\rho'_m)\,\mathrm{d}\rho', \quad k,m = 1,\ldots,n \\
&B^{(3)}_{km} = \int_{\rho_k}^{\rho_{k+1}} R_2(\rho_m,\rho')\,\mathrm{d}\rho', \quad B^{(4)}_{km} = \int_{\rho_k}^{\rho_{k+1}} \rho' R_4(\rho',\rho_m)\,\mathrm{d}\rho', \quad k = 1,\ldots,n;\ m = 1,\ldots,n+1
\end{aligned}
\tag{38}
$$

This system of $2n+2$ equations is sufficient to determinate $2n+2$ unknowns $h^*(\rho'_i)$, $p^*(\rho'_i)$, $i = 1,\ldots,n$ and constants δ^*, P_{H}/P.

Knowing the distribution $h^*(\rho'_i)$ the deflexion in the excavitation zone (31) can be calculated as

$$
w^*(\rho_i) = \kappa_2\sum_{k=i}^{n} h^*(\rho'_k)B^{(0)}_{km}, \quad i = 1,\ldots,n+1 \tag{39}
$$

where

$$
B^{(0)}_{ik} = \int_{\rho_k}^{\rho_{k+1}} \frac{\mathrm{d}\rho'}{\sqrt{\rho'^2 - \rho_i^2}}, \quad i = 1,\ldots,n+1;\ k = i,\ldots,n
$$

The numerical analysis was performed to display the influence of the input dimensionless parameters ϑ, κ, κ_2 on the distribution of the contact pressure $p^*(\rho)$, deflexion in the excavitation zone $w^*(\rho)$, ratio P_{H}/P and centre displacement δ^*.

It was disclosed that the excavitation has an influence on the contact pressure for thin ($\kappa_2 > 1$) layer only. The distribution of the function $p^*(\rho)$ in the contact zone for some values of the Winkler medium stiffness ϑ and fixed parameter $\kappa = 1$, $\kappa_2 = 2$ is presented in Fig. 2a. By the dotted line the well-known result (Johnson, 1987) for an elastic half-space

$$
p^*(\rho) = \frac{3}{2\pi}\sqrt{1-\rho^2} \tag{40}
$$

is shown for a comparision. Generally, the excavitation causes the decreasing of the contact pressure in the centre of contact zone. The result for $\vartheta = 100$ is close to that in the contact problem for an elastic layer resting on a rigid smooth base (Li and Dempsey, 1990).

The dependence of parameters P_{H}/P and δ^* with the dimensionless stiffness ϑ is shown in Fig. 3 for some values of the layer thickness κ_2 and fixed ratio $\kappa = 1$. The value P_{H}/P and displacement δ^* decrease for the rising of the stiffness of the Winkler medium. This behaviour is more strong for a thin layer. Analysis of the ratio P_{H}/P permits to make the conclusion that the load P, needed to reach contact area $l = l_{\mathrm{H}}$, decreases for a empty excavitation. Thus, the general trend is that the excavitation yields the greater contact region.

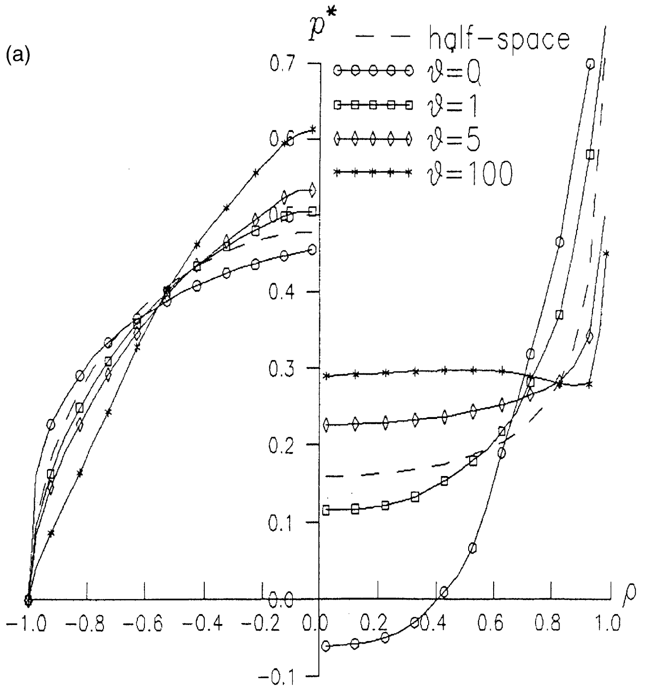

Fig. 2. Contact pressure distributions.

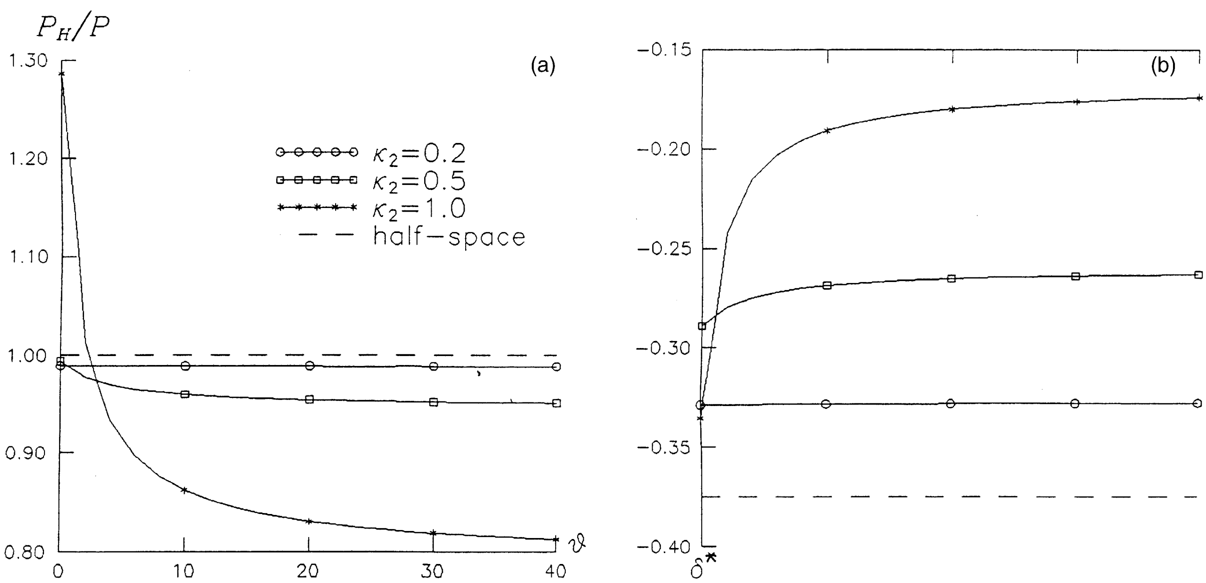

Fig. 3. Dependence of parameters P_H/P and δ^* with the stiffness ϑ.

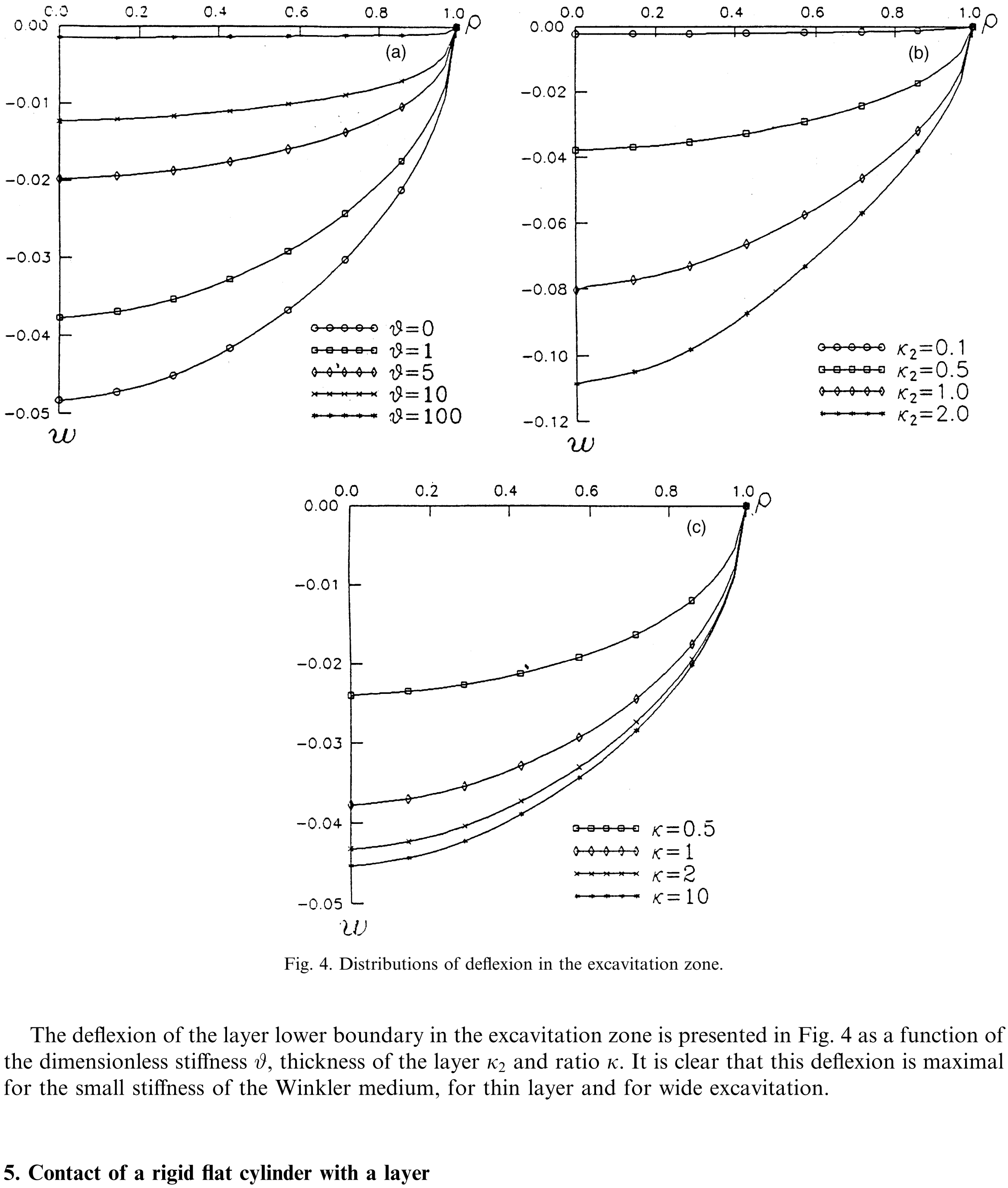

Fig. 4. Distributions of deflexion in the excavitation zone.

The deflexion of the layer lower boundary in the excavitation zone is presented in Fig. 4 as a function of the dimensionless stiffness ϑ, thickness of the layer κ_2 and ratio κ. It is clear that this deflexion is maximal for the small stiffness of the Winkler medium, for thin layer and for wide excavitation.

5. Contact of a rigid flat cylinder with a layer

In second example the punch is considered as a rigid flat cylinder of radius l (Fig. 1c) and function $g^*(\rho)$ can be written as

$$g^*(\rho) = 0 \tag{41}$$

Then the system of integral equations (26) of the problem under consideration has the following form

$$\begin{aligned} &h^*(\tau) - \frac{2\kappa_2}{\pi}\int_0^1 h^*(\tau')R_1^*(\tau',\tau)\,\mathrm{d}\tau' + \frac{1}{\pi}\int_0^1 \rho' q^*(\rho')R_2^*(\rho',\tau)\,\mathrm{d}\rho' = -p_0 Q_1(\tau), \quad 0 \leqslant \tau \leqslant 1 \\ &2\kappa_1\kappa_2\int_0^1 h^*(\tau')R_3^*(\tau',\rho)\,\mathrm{d}\tau' - \kappa_1\int_0^1 \rho' p^*(\rho')R_4^*(\rho',\rho)\,\mathrm{d}\rho' = \delta^*, \quad 0 \leqslant \rho \leqslant 1 \\ &2\pi\int_0^1 \rho' p^*(\rho')\,\mathrm{d}\rho' = 1 \end{aligned} \tag{42}$$

Note that in this problem the radius of the contact area is given and equal to the cylinder radius l. The geometrical singularity at the edge of punch yields that the contact pressure is singular as $\rho \to 1$. To display this fact let us present the function $p^*(\rho)$ in the form

$$p^*(\rho) = \frac{p_0}{\sqrt{1-\rho^2}} + q^*(\rho), \quad 0 \leqslant \rho \leqslant 1 \tag{43}$$

where the regular function $q^*(\rho)$ is a new unknown and the constant p_0, which can be treated as a stress intensity factor at the punch edge, is also unknown.

Substituting the presentation (43) into (42) we arrive at the close system of integral equations for unknowns $h^*(\rho)$, $q^*(\rho)$, p_0 and δ^*

$$\begin{aligned} &h^*(\tau) - \frac{2\kappa_2}{\pi}\int_0^1 h^*(\tau')R_1^*(\tau',\tau)\,\mathrm{d}\tau' + \frac{1}{\pi}\int_0^1 \rho' q^*(\rho')R_2^*(\rho',\tau)\,\mathrm{d}\rho' = -p_0 Q_1(\tau), \quad 0 \leqslant \tau \leqslant 1 \\ &2\kappa_1\kappa_2\int_0^1 h^*(\tau')R_3^*(\tau',\rho)\,\mathrm{d}\tau' - \kappa_1\int_0^1 \rho' q^*(\rho')R_4^*(\rho',\rho)\,\mathrm{d}\rho' = \delta^* + p_0 Q_2(\rho), \quad 0 \leqslant \rho \leqslant 1 \\ &2\pi\int_0^1 \rho' q^*(\rho')\,\mathrm{d}\rho' = 1 - 2\pi p_0 \end{aligned} \tag{44}$$

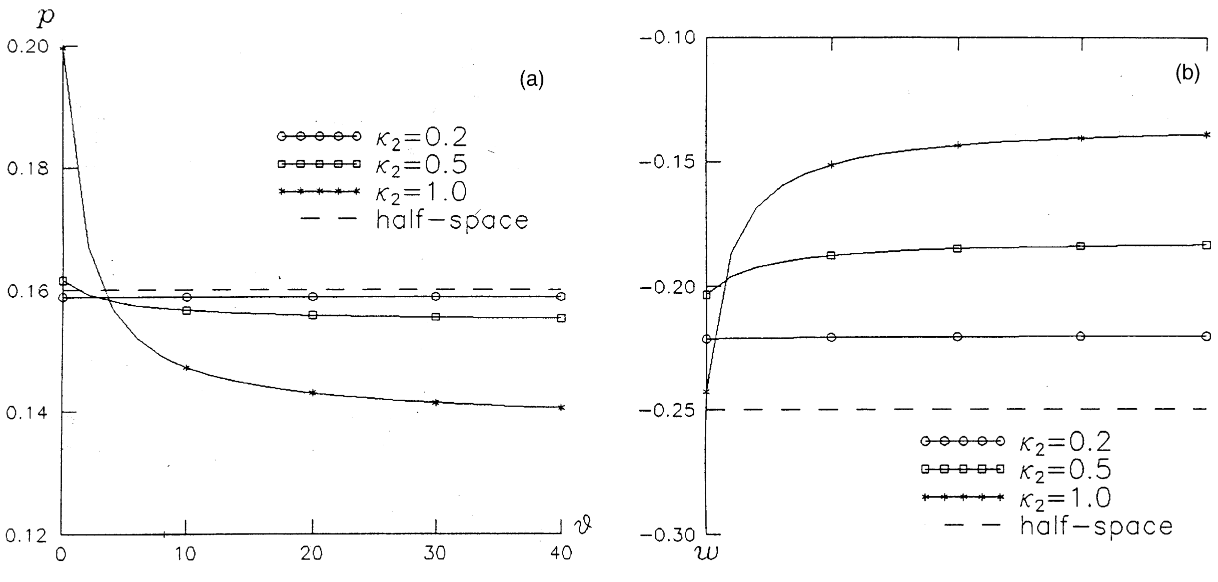

Fig. 5. The dependence of parameters p_0 and δ^* with the stiffness ϑ.

where

$$Q_1(\tau) = \frac{1}{\pi\kappa_1}\int_0^\infty \beta^{-1}F_2^*(\beta)\sin(\kappa_1\beta)\sin(\kappa_2\beta\tau)\,\mathrm{d}\beta$$

$$Q_2(\rho) = \int_0^\infty \beta^{-1}F_3^*(\beta)\sin(\kappa_1\beta)J_0(\kappa_1\beta\rho)\,\mathrm{d}\beta$$

Here the integrals (Gradshteyn and Ryzhik (1965))

$$\int_0^1 \frac{\rho\,\mathrm{d}\rho}{\sqrt{1-\rho^2}} = 1, \qquad \int_0^1 \frac{\rho J_0(a\rho)\,\mathrm{d}\rho}{\sqrt{1-\rho^2}} = a^{-1}\sin(a)$$

are used.

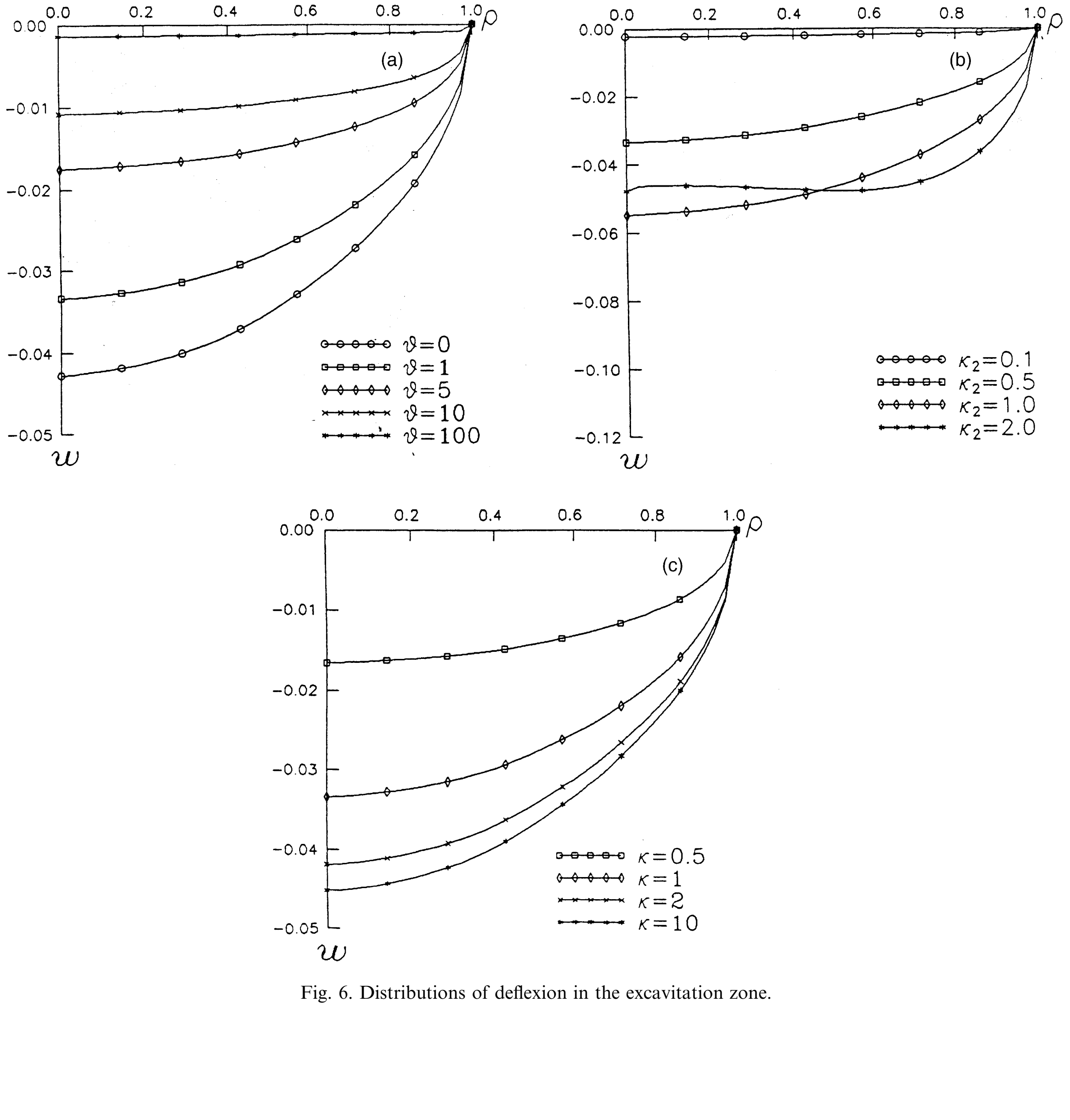

Fig. 6. Distributions of deflexion in the excavitation zone.

Similarly to (37), we obtain the discretization scheme for Eq. (44) in the following form

$$\begin{aligned}
&h^*(\rho'_m) - \frac{2\kappa_2}{\pi}\sum_{k=1}^{n} h^*(\rho'_k)B^{(1)}_{km} + \frac{1}{\pi}\sum_{k=1}^{n} q^*(\rho'_k)B^{(2)}_{km} + p_0 Q_1(\rho'_m) = 0, \quad m = 1,\ldots,n \\
&2\kappa_1\kappa_2\sum_{k=1}^{n} h^*(\rho'_k)B^{(3)}_{km} - \kappa_1\sum_{k=1}^{n} q^*(\rho'_k)B^{(4)}_{km} - \delta^* - p_0 Q_2(\rho_m) = 0, \quad m = 1,\ldots,n+1 \\
&\frac{\pi}{n^2}\sum_{k=1}^{n}(2k-1)q^*(\rho'_k) + 2\pi p_0 Q_0 = 1
\end{aligned} \tag{45}$$

This system of $2n+2$ linear algebraic equations is sufficient to determinate $2n+2$ unknowns $h^*(\rho'_i)$, $q^*(\rho'_i)$, $i = 1,\ldots,n$ and constants δ^*, p_0.

Knowing the distributions $h^*(\rho'_i)$, $q^*(\rho'_i)$ and p_0 the deflexion $w^*(\rho_i)$ can be calculated from formula (39) and the contact pressure from that (43).

The distribution of the contact pressure is presented in Fig. 2b for various values of the dimensionless stiffness ϑ and $\kappa = 1$, $\kappa_2 = 2$. For stiff Winkler medium ($\vartheta = 100$) the result is closed to the solution of the contact problem for an elastic layer resting on a rigid base (Li and Dempsey, 1990). Decreasing of the stiffness yields the falling of the contact pressure in centre of the contact area. For $\vartheta = 0$ (hollow excavitation) the separation of the contact zone is observed. This result is observed for a thin layer only. The Hertz distribution

$$p^*(\rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} \tag{46}$$

is presented in Fig. 2b by the dotted line.

Fig. 5 shows the dependences of parameters p_0 and δ^* with the Winkler medium stiffness. The deflexion in the excavitation zone $w^*(\rho)$ in the dependence of the parameters ϑ, κ, κ_2 is presented in Fig. 6. The numerical results obtained for the flat cylinder are in general similarity with those for the rigid sphere. Some differences presented in Fig. 6b for the wide excavitation ($\kappa_2 = 2.0$) are explained by the contact region separation, which takes place in this case.

Appendix A

Let the function $g(\zeta)$ decays exponentially for $\zeta \to \infty$. We approximate this function by a function $\hat{g}(\zeta)$, which is defined as

$$\hat{g}(\zeta) = \sum_{m=1}^{M} a_m e^{-m\beta\zeta} \tag{A.1}$$

where the constants a_m, $m = 1,\ldots,M$ and are determinated by the summation of the squared error

$$S = \sum_{l=1}^{L}\left[g(\zeta_l) - \hat{g}(\zeta_l)\right]^2 \tag{A.2}$$

The minimizing conditions

$$\frac{\partial S}{\partial a_k} = 0, \quad k = 1,\ldots,M \tag{A.3}$$

gives the system of equations for the determination of the constants a_m, $m = 1,\ldots,M$

$$\sum_{m=1}^{M} a_m \left\{ \sum_{l=1}^{L} \exp[-(m+k)\beta\zeta_l] \right\} = \sum_{l=1}^{L} g(\zeta_l)\exp(-k\beta\zeta_l), \quad k = 1,\ldots,M \tag{A.4}$$

which can be simplified if the points ζ_l, $l = 1,\ldots,L$ are chosen as

$$\zeta_l = l\zeta_{\max}/L, \quad l = 1,\ldots,L \tag{A.5}$$

Thus

$$\sum_{l=1}^{L} \exp[-(m+k)\beta\zeta_l] = \sum_{l=1}^{L} a_{mk}^{l} = \frac{1 - a_{mk}^{L+1}}{1 - a_{mk}}, \quad k = 1,\ldots,M \tag{A.6}$$

where

$$a_{mk} = \exp[-(m+k)\beta\zeta_{\max}/L] \tag{A.7}$$

and the system (A.4) can be transformed

$$\sum_{m=1}^{M} a_m \left\{ \frac{1 - a_{mk}^{L+1}}{1 - a_{mk}} \right\} = \sum_{l=1}^{L} g(\zeta_l)\exp(-k\beta\zeta_l), \quad k = 1,\ldots,M \tag{A.8}$$

The constant β is determinated iteratively from the condition of the value S minimum. The accuracy of this method is provided by the choice of constants M, L and $\zeta_{\max}$.

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